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On a Conjecture of Cameron and Liebler

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Cameron–Liebler line classes arose from an attempt by Cameron and Liebler to classify those collineation groups of $PG(n, q)$ which have the same number of orbits on points as on lines. They satisfy several equivalent properties; among them, constant intersection with spreads. Cameron and Liebler conjectured that, apart from some 'obvious' examples, no sets of lines of this type exist in $PG(3, q)$. This paper introduces a connection between Cameron–Liebler line classes in $PG(3, q)$ and blocking sets in $PG(2, q)$, and uses it to provide the strongest results to date concerning the non-existence of certain of these sets. In addition, a complete classification of Cameron–Liebler line classes in $PG(3, 3)$ is obtained, with the main result being that there is, essentially, a unique counterexample to Cameron and Liebler's conjecture in this space.

1. INTRODUCTION

Let $\Sigma = PG(3, q)$ denote the three-dimensional projective space over the finite field of q elements. Let V (resp. W) denote the $\mathbb{Q}$-vector space of all functions from the point set (resp. line set) of Σ to $\mathbb{Q}$. In their paper [8], Cameron and Liebler introduce the concept of Cameron–Liebler line classes (calling them 'special' line classes). These are sets of lines whose characteristic functions are in the image of the map $\alpha : V \to W$ given with respect to standard bases of V and W by a point–line incidence matrix of Σ. Cameron and Liebler, and Penttila in [11], prove that this property is equivalent to many other interesting geometric properties. Before stating the theorem giving these properties, we introduce some notation.

Following Penttila, we define a *clique* of $PG(3, q)$ to be a set $star(P)$ of all lines on a point P, or $line(\pi)$ of all lines in a plane π. Then any clique forms a projective plane $PG(2, q)$ with 'points', the lines of the clique and 'lines', the plane pencils of the clique. For an incident point-plane pair (P, π), we will denote the plane pencil they define by $pen(P, \pi)$. The following theorem gives the properties currently known to be satisfied by Cameron–Liebler line classes.

THEOREM 1.1 (CAMERON, LIEBLER, PENTTILA). *Let $\mathcal{L}$ be a set of lines of $PG(3, q)$ with characteristic function $\chi_{\mathcal{L}}$. Then the following are equivalent:*

(1) *$\chi_{\mathcal{L}}$ is in the range of α.*
(2) *$\chi_{\mathcal{L}}$ is orthogonal to the kernel of α^t.*
(3) *There exists $x > 0$ such that $|\mathcal{L} \cap \mathcal{S}| = x$ for all spreads $\mathcal{S}$ of $PG(3, q)$.*
(4) *There exists $x > 0$ such that $|\mathcal{L} \cap \mathcal{S}| = x$ for all regular spreads $\mathcal{S}$ of $PG(3, q)$.*
(5) *For every regulus $\mathcal{R}$ of $PG(3, q)$, $\mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \mathcal{R}^{opp}$.*
(6) *There exists x such that for any incident point-plane pair (P, π) of Σ*

$$|star(P) \cap \mathcal{L}| + |line(\pi) \cap \mathcal{L}| = x + (q + 1)|pen(P, \pi) \cap \mathcal{L}|. \tag{1}$$

(7) *There exists x such that for any line l of Σ*

$$|\{m \in \mathcal{L} : m \text{ meets } l, m \neq l\}| = (q + 1)x + (q^2 - 1)\chi_{\mathcal{L}}(l). \tag{2}$$

(8) *There exists x such that for any skew lines l and m,*

$$|\{n \in \mathcal{L} : n \text{ is a transversal to } l \text{ and } m\}| = x + q(\chi_{\mathcal{L}}(l) + \chi_{\mathcal{L}}(m)). \tag{3}$$

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It follows from the proof of the theorem that the number x is the same for each of the conditions. This number is called the *parameter* of the Cameron–Liebler line class. It was proven in [8] that a Cameron–Liebler line class of parameter x has exactly $x(q^2+q+1)$ elements. As there are q^2+1 lines in any spread of $\Sigma = PG(3,q)$, we have $1 \leq x \leq q^2$. Sets of this type are interesting from many perspectives; see [4, 7, 8, 11] and [9] for discussions.

In [8], it was proven that the cliques are exactly the Cameron–Liebler line classes of parameter one and that unions of two disjoint cliques are exactly the classes of parameter two; and conjectured that, apart from these examples and their complements (of parameter q^2 and q^2-1), no sets of this type exist. In [11], it was proven that for $q > 2$ no Cameron–Liebler line class of parameter three or four exists (apart from the possibility of a class of parameter four in $PG(3,q)$ with $q = 3$ or 4). In a recent joint paper [4] with A. A. Bruen, it was proven using a counting argument that no Cameron–Liebler line classes exist in $PG(3,q)$ for $2 < x \leq \sqrt{q}$. In Section 2 of this paper, we demonstrate a connection between Cameron–Liebler line classes in $PG(3,q)$ and blocking sets in $PG(2,q)$ which allows us to improve significantly on this last result, especially when $q = p$ is prime. In this case, we prove that no Cameron–Liebler line class exists for $2 < x \leq \frac{1}{2}(p+1)$. Then, in Section 3 we give a counterexample to Cameron and Liebler's conjecture by exhibiting a Cameron–Liebler line class of parameter five in $PG(3,3)$; that is, a set of 65 lines of $PG(3,3)$ which contains exactly half of every spread of $PG(3,3)$. We also show that this line class is essentially the only counterexample to Cameron and Liebler's conjecture in $PG(3,3)$. The existence of this line class is derived by ad hoc methods; this is possible because the number of points in $PG(3,3)$ is quite small. However, the line class is found to have a very precise structure (see Theorem 3.2); and this suggests that a similar class can be constructed in $PG(3,q)$ for all prime powers q. In fact, such a construction has been carried out by the author and A. A. Bruen and will appear in [5].

2. A CONNECTION WITH BLOCKING SETS

Throughout the discussion, $q > 2$ will be a fixed prime power. Let $s(= s(q))$ be the smallest integer for which there exists a blocking set of size s in $PG(2,q)$ (a blocking set is a set of points which intersects every line but contains no line). Let $\mathcal{L}$ be a Cameron–Liebler line class of parameter x in $\Sigma = PG(3,q)$ and assume that there exists no line class of parameter $x-1$ in Σ. This implies that for any clique $\mathcal{C}$ of Σ, $|\mathcal{C} \cap \mathcal{L}| < q^2+q+1$ (that is, $\mathcal{C} \not\subset \mathcal{L}$) because if $\mathcal{C} \subset \mathcal{L}$ then $\mathcal{L} \setminus \mathcal{C}$ would be a line class of parameter $x-1$. As the Cameron–Liebler line classes of parameter $x \leq 2$ are classified, we can also assume that $x > 2$. The theorem that we will prove depends upon the following observation:

LEMMA 2.1. *If $\mathcal{C}$ is any clique of Σ and if $x < |\mathcal{C} \cap \mathcal{L}| \leq x+q$ then the lines of $\mathcal{C} \cap \mathcal{L}$ form a blocking set in $\mathcal{C}$.*

PROOF. Let $P \in \Sigma$ be any point, and π a plane on P. Put $M = |star(P) \cap \mathcal{L}|$, $N = |line(\pi) \cap \mathcal{L}|$, and $Q = |pen(P,\pi) \cap \mathcal{L}|$. By Eqn. (1) we have that $N = x + (q+1)Q - M$ so that if $x < M \leq x+q$, $Q = 0$ implies $N = x - M < 0$, and $Q = q+1$ implies $N = x + q^2 + 2q + 1 - M \geq q^2+q+1$. But neither of these possibilities can occur—the assumption that there exists no Cameron–Liebler line class of parameter $x-1$ implies that $N < q^2+q+1$, and $N < 0$ is clearly absurd; and therefore we conclude that the lines of $star(P) \cap \mathcal{L}$ form a blocking set in the projective plane $star(P)$. The case where $x < N \leq x+q$ is proven in an identical fashion. $\square$

THEOREM 2.1. *If $2 < x < s-q$ then there exists no Cameron–Liebler line class of parameter x in $\Sigma = PG(3,q)$.*

PROOF. It is proven in [11] that there exists no Cameron–Liebler line class of parameter three in Σ if $q > 2$. Hence, we may assume inductively that there exists no Cameron–Liebler line class of parameter $x - 1$ in Σ. Now, if x satisfies the inequalities given in the hypothesis, then the above lemma implies that for any clique $\mathcal{C}$ of Σ, $x < |\mathcal{C} \cap \mathcal{L}| \leq x + q$ is impossible as $x + q < s$. Now assume that $0 < |\mathcal{C} \cap \mathcal{L}| \leq x$ and for convenience that $\mathcal{C} = star(P)$ for some $P \in \Sigma$ (the case where $\mathcal{C} = line(\pi)$ is identical). As we certainly have $s \leq \frac{3}{2}(q+1)$ (see [1]) and $q > 2$, $x + q < s$ implies $x < q + 1$, so there exists a tangent to the set $star(P) \cap \mathcal{L}$; that is, a plane π on P such that $|pen(P, \pi) \cap \mathcal{L}| = 1$. For this π, and with M and N defined as above, we have $M + N = x + q + 1$ by (1). But as $0 < M \leq x$ this implies that $q + 1 \leq N \leq x + q$, and as $x < q + 1$ this contradicts Lemma 2.1. Hence, $0 < |\mathcal{C} \cap \mathcal{L}| < x + q$ cannot occur. But now if for some clique $\mathcal{C}$, $|\mathcal{C} \cap \mathcal{L}| = 0$, then (again assuming $\mathcal{C} = star(P)$) for any plane π on P, (1) implies that $|line(\pi) \cap \mathcal{L}| = x$, so in fact $0 \leq |\mathcal{C} \cap \mathcal{L}| < x + q$ is impossible.

Now let $l \notin \mathcal{L}$ be a line. Then by Eqn. (2), exactly $x(q+1)$ lines of $\mathcal{L}$ intersect l. This means that if the points of l are $P_1, \ldots, P_{q+1}$, we have

$$\sum_{i=1}^{q+1} |star(P_i) \cap \mathcal{L}| = x(q+1),$$

and hence for some $P_i \in l$, $|star(P_i) \cap \mathcal{L}| \leq x$, contradicting the above. Hence, $2 < x < s - q$ is impossible. □

COROLLARY 1. *If p is an odd prime then there exists no Cameron–Liebler line class of parameter $2 < x \leq \frac{1}{2}(p+1)$ in $PG(3, p)$.*

PROOF. In [1], Blokhuis proved that the smallest blocking set in $PG(2, p)$ for p an odd prime has size $\frac{3}{2}(p+1)$. Therefore, by the above theorem, there exists no Cameron–Liebler line class of parameter x in $PG(3, p)$ for $2 < x < \frac{3}{2}(p+1) - p = \frac{1}{2}(p+3)$. □

COROLLARY 2. *If $q > 7$ is not a square and not equal to 27, then there exists no Cameron–Liebler line class of parameter $2 < x < \sqrt{2q} + 1$ in $PG(3, q)$.*

PROOF. It was proven by Blokhuis and Brouwer in [2] and independently by Bruen and Silverman in [6] that if q satisfies the assumptions above then the size of a minimal blocking set in $PG(2, q)$ is at least $q + \sqrt{2q} + 1$. □

REMARK . In the case where q is a square, it is well known (see [3]) that the minimal size of a blocking set in $PG(2, q)$ is $q + \sqrt{q} + 1$, and that this bound is achieved by Baer subplanes. Thus, in this case, the bound given by Theorem 2.1 is essentially the same as that given in [4].

3. CAMERON–LIEBLER LINE CLASSES IN $PG(3, 3)$

We now study Cameron–Liebler line classes of parameter x, $2 < x < q^2 - 1$, in $\Sigma = PG(3, 3)$. If $\mathcal{L}$ is a Cameron–Liebler line class of parameter x in $PG(3, q)$, then $\mathcal{L}^c$ (all lines of $PG(3, q)$ not in $\mathcal{L}$) is a line class of parameter $q^2 + 1 - x$, so we can assume without loss of generality that our line classes have parameter $x \leq 5$. Also, as Penttila (in [11]) has proven that no line class of parameter three exists in $PG(3, q)$ if $q > 2$, we can, in fact, limit our study to the case where $x = 4$ or 5.

3.1. Parameter $x = 4$. In this section we prove that there are no Cameron–Liebler line classes of parameter four in $PG(3,3)$. Assume that $\mathcal{L}$ is such a set.

LEMMA 3.1. $|\mathcal{L} \cap \mathcal{C}|$ *cannot be odd.*

PROOF. If $|\mathcal{L}\cap\mathcal{C}| = 13$, then $\mathcal{L} \setminus \mathcal{C}$ would be a line class of parameter three. If $|line(\pi)\cap\mathcal{L}| = 11$, then there exists some $P \in \pi$ such that $|pen(P,\pi) \cap \mathcal{L}| = 2$, so by (1) we would have $|star(P) \cap \mathcal{L}| = 1$, contradicting the fact that $star(P) \supset pen(P,\pi)$. If $|\mathcal{L} \cap \mathcal{C}| = 5$, then by Lemma 2.1, the lines of $\mathcal{C}$ form a blocking set in $\mathcal{C} \equiv PG(2,3)$, but minimal blocking sets in this plane have size six. Thus, $|\mathcal{C} \cap \mathcal{L}| = 5$ is impossible. If $|line(\pi) \cap \mathcal{L}| = 3$, there exists some $P \in \pi$ with $|pen(P,\pi) \cap \mathcal{L}| = 1$, so $M = 1 + 4(1) = 5$, contradiction. Similarly, $|line(\pi) \cap \mathcal{L}| = 1$ gives a contradiction by choosing $P \in \pi$ with $|pen(P,\pi) \cap \mathcal{L}| = 0$; and it is now easy to see using condition (1) and examining possibilities for $|pen(P,\pi) \cap \mathcal{L}|$ that neither $|\mathcal{L} \cap \mathcal{C}| = 7$ nor $|\mathcal{L} \cap \mathcal{C}| = 9$ can occur. □

LEMMA 3.2. $|\mathcal{L} \cap \mathcal{C}| \neq 0, 4, 8$ *or* 12.

PROOF. Assume that $|star(P)\cap\mathcal{L}| = 8$. Then $|star(P)\cap\mathcal{L}^c| = 5 > 4$, so there exist three lines of $star(P) \cap \mathcal{L}^c$ which are contained in a pencil $pen(P,\pi)$, so $|pen(P,\pi) \cap \mathcal{L}| \leq 1$. If $|pen(P,\pi) \cap \mathcal{L}| = 0$, then by Eqn. (1) we have that $8 + |line(\pi) \cap \mathcal{L}| = 4$ which is absurd, so it must be that $|pen(P,\pi) \cap \mathcal{L}| = 1$. Now $|pen(P,\pi) \cap \mathcal{L}| \leq |line(\pi) \cap \mathcal{L}|$, so Eqn. (1) implies that

$$8 = |star(P) \cap \mathcal{L}| \leq x + q|pen(P,\pi) \cap \mathcal{L}| \leq 4 + 3 = 7.$$

Therefore, $|\mathcal{L} \cap \mathcal{C}| \neq 8$.

If $|star(P) \cap \mathcal{L}| = 12$, then there exists some plane $\pi \ni P$ such that $|pen(P,\pi) \cap \mathcal{L}| = 4$, and then we have that $|line(\pi) \cap \mathcal{L}| = 8$ by Eqn. (1), a contradiction. Similarly, $|\mathcal{L} \cap \mathcal{C}| = 4$ or 0 give contradictions. □

Therefore, the only three possibilities for $|\mathcal{L} \cap \mathcal{C}|$ are two, six, and ten.

LEMMA 3.3. *Let l be any line of Σ. If $l \in \mathcal{L}$, the four numbers $|star(P) \cap \mathcal{L}|$ for $P \in l$ are either* $\{10, 10, 6, 2\}$ *or* $\{10, 6, 6, 6\}$. *If $l \notin \mathcal{L}$, the four numbers $|star(P) \cap \mathcal{L}|$ for $P \in l$ are either* $\{10, 2, 2, 2\}$ *or* $\{6, 6, 2, 2\}$.

PROOF. As $\mathcal{L}$ is a Cameron–Liebler line class of parameter four in $PG(3,3)$, any line in $\mathcal{L}$ intersects exactly $q^2 + xq + x = 25$ lines of $\mathcal{L}$ and any line not in $\mathcal{L}$ intersects exactly $xq + x = 16$ lines of $\mathcal{L}$ by Eqn. (2). This means that the four numbers $|star(P) \cap \mathcal{L}|$ for $P \in l$ must sum to either 28 or 16, depending on whether $l \in \mathcal{L}$ or not. As the only numbers possible for $|star(P) \cap \mathcal{L}|$ are 2, 6, and 10, the result follows. □

Recall that a *cap* in Σ is a set of points where no three are collinear; a cap is *complete* if it is not contained in a larger cap.

LEMMA 3.4. *The points $P \in \Sigma$ with $|star(P) \cap \mathcal{L}| = 10$ form a complete cap $\mathcal{K}$ of size eight in Σ.*

PROOF. By the previous lemma, it is clear that the points of $\mathcal{K}$ form a cap in Σ. We also see from this lemma that each line in $\mathcal{L}$ must contain at least one of these points, and that any line which contains two must be in $\mathcal{L}$. Thus, if the number of points P such that $|star(P)\cap\mathcal{L}| = 10$ is a, we must have

$$10 \times a - \binom{a}{2} = |\mathcal{L}| = 52,$$

which gives a quadratic equation in a with solutions $a = 8$ or $a = 13$. But any cap in Σ has at most 10 points (see, for example, [10]), so it must be that $a = 8$. Hence, the set of points forms a cap of size 8. The completeness of this cap will follow from the next lemma. $\square$

Given a line l of Σ, we will call it a line of type $(6, 6, 2, 2)$ if two of its points have $|star(P) \cap \mathcal{L}| = 6$ and two of its points have $|star(P) \cap \mathcal{L}| = 2$, et cetera.

LEMMA 3.5. *Let $P \in \Sigma$.*

(1) *If $|star(P) \cap \mathcal{L}| = 2$, then the two lines of $star(P) \cap \mathcal{L}$ are of type $(10, 10, 6, 2)$ and of the remaining eleven lines, four are of type $(10, 2, 2, 2)$ and seven are of type $(6, 6, 2, 2)$.*
(2) *If $|star(P) \cap \mathcal{L}| = 6$, then two of the lines of $star(P) \cap \mathcal{L}$ are of type $(10, 10, 6, 2)$ and four are of type $(10, 6, 6, 6)$, and the seven lines of $star(P)$ not in $\mathcal{L}$ are of type $(6, 6, 2, 2)$.*
(3) *If $|star(P) \cap \mathcal{L}| = 10$, then seven of the lines of $star(P) \cap \mathcal{L}$ are of type $(10, 10, 6, 2)$ and three are of type $(10, 6, 6, 6)$, and the three lines of $star(P)$ not in $\mathcal{L}$ are of type $(10, 2, 2, 2)$.*

PROOF. We prove the first assertion; the other two are similar. If $|star(P) \cap \mathcal{L}| = 2$, then the two lines of $\mathcal{L}$ on P are of type $(10, 10, 6, 2)$ as the other type of line in $\mathcal{L}$ does not contain a point with $|star(P) \cap \mathcal{L}| = 2$. These two lines account for four points with $|star(P) \cap \mathcal{L}| = 10$, so because there are eight of these points in total, the lines not in $\mathcal{L}$ on P must account for the other four. Thus, exactly four of them must have type $(10, 2, 2, 2)$ while the other seven have type $(6, 6, 2, 2)$. $\square$

Now Lemma 3.5 implies that the cap from Lemma 3.4 is complete; in fact the lemma implies that each point of $\Sigma \setminus \mathcal{K}$ is on exactly two secants to $\mathcal{K}$. We now prove that this is impossible.

LEMMA 3.6. *If $\mathcal{K}$ is an 8-cap of Σ, then there exists some point of $\Sigma \setminus \mathcal{K}$ which is on only one secant to $\mathcal{K}$.*

PROOF. There are exactly 28 secant lines to $\mathcal{K}$, each of which contains two points which are not in $\mathcal{K}$. Thus, there are 56 flags (P, l) where $P \notin \mathcal{K}$ and l is a secant to $\mathcal{K}$. On the other hand, if each of the thirty-two points of $\Sigma \setminus \mathcal{K}$ were on at least two secants to $\mathcal{K}$, there would be at least 64 such flags. $\square$

THEOREM 3.1. *There exists no Cameron–Liebler line class of parameter four in $PG(3, 3)$.*

PROOF. This follows immediately from the above lemma and the remarks preceeding it. $\square$

3.2. Parameter $x = 5$. Now we will demonstrate the falsehood of Cameron and Liebler's conjecture by exhibiting a line class of parameter five in $PG(3, 3)$. It will also follow from the analysis that any line class of parameter five in this space is essentially identical to the constructed class. Let $\mathcal{L}$ be a Cameron–Liebler line class of parameter five in Σ. As $\mathcal{L}^c$ (the set of lines of Σ not in $\mathcal{L}$) is also a line class of parameter five, if it can be shown that $|\mathcal{L} \cap \mathcal{C}| = k$ is impossible, it follows that $|\mathcal{L} \cap \mathcal{C}| = 13 - k$ is impossible as well. If $|\mathcal{L} \cap \mathcal{C}| = 13$ for some clique $\mathcal{C}$, then $\mathcal{L} \setminus \mathcal{C}$ is a line class of parameter four, which by the previous section is impossible, so neither $|\mathcal{L} \cap \mathcal{C}| = 13$ nor $|\mathcal{L} \cap \mathcal{C}| = 0$ can occur.

LEMMA 3.7. $|\mathcal{L} \cap \mathcal{C}| \neq 1, 4, 5, 8, 9$ *or* 12.

PROOF. The impossibility of $|\mathcal{L} \cap \mathcal{C}| = 5$ follows from Lemma 2.1 as in the $x = 4$ case. If $|star(P) \cap \mathcal{L}| = 1$, then for some $\pi \ni P$ we have $|pen(P, \pi) \cap \mathcal{L}| = 1$, so $1 + |line(\pi) \cap \mathcal{L}| = 5 + 4$ or $|line(\pi) \cap \mathcal{L}| = 8$, a contradiction. If $|star(P) \cap \mathcal{L}| = 4$ and for some $\pi \ni P$ we have $|pen(P, \pi) \cap \mathcal{L}| \geq 3$, then $4 + |line(\pi) \cap \mathcal{L}| \geq 5 + 4 \times 3$, so $|line(\pi) \cap \mathcal{L}| \geq 13$. Hence, the lines of $|star(P) \cap \mathcal{L}|$ form an arc in the $PG(2, 3)$ formed by $star(P)$, so there exists a plane $\pi \ni P$ such that $|pen(P, \pi) \cap \mathcal{L}| = 0$, and then $|line(\pi) \cap \mathcal{L}| = 1$, contradicting the above. $\square$

Now we note that 10 and 11 cannot both occur as $|star(P) \cap \mathcal{L}|$ for $P \in \Sigma$: if they did, let l be the line which joins the two points. Then $l \in \mathcal{L}$ as otherwise there would be at least 21 lines of $\mathcal{L}$ on l, but by Eqn. (2) of Theorem 1.1, there are exactly 20 lines of $\mathcal{L}$ on any line not in $\mathcal{L}$. Therefore, by the same condition, there are exactly 28 lines of $\mathcal{L} \setminus \{l\}$ on l, so that

$$\sum_{P \in l} |star(P) \cap \mathcal{L}| = 32. \tag{4}$$

Thus, we would have $a + b + 10 + 11 = 32$ with a and b in $\{2, 3, 6, 7, 10, 11\}$, and this cannot occur. Hence, 10 and 11 cannot both occur as $|star(P) \cap \mathcal{L}|$. Assume that 11 occurs, so that 10 does not. By considering possible sums of the form $a + b + c + d = 32$ with $a, b, c, d \in \{2, 3, 6, 7, 11\}$, it is easily seen that no line in $\mathcal{L}$ can contain a point P with $|star(P) \cap \mathcal{L}| = 2$ or 6, and therefore (as every point of Σ is on at least one line of $\mathcal{L}$) only the numbers 3, 7 and 11 can occur as intersection numbers of stars with $\mathcal{L}$. Using Eqn. (1) it is then easy to show that only 2, 6 and 10 can occur as $|line(\pi) \cap \mathcal{L}|$ for π a plane of Σ.

An argument identical to that used in Lemma 3.4 shows that the points with $|star(P) \cap \mathcal{L}| = 11$ form a cap of size 10, but as $10 = 3^2 + 1$ such a cap must be an elliptic quadric $\mathcal{O}$ (see [10]). Moreover, as there are exactly nine secants to an elliptic quadric at any of its points, there must be exactly two tangent lines to $\mathcal{O}$ at each point P which are in $\mathcal{L}$, and two which are not. Note that the complement of such a set $\mathcal{L}$ will consist of all external lines to $\mathcal{O}$ together with half of the tangent lines to $\mathcal{O}$ at each point (this corresponds to replacing the assumption that $|star(P) \cap \mathcal{L}| = 11$ for some point with $|star(P) \cap \mathcal{L}| = 10$ for some point). Therefore, we have proven the following.

THEOREM 3.2. *If $\mathcal{L}$ is a Cameron–Liebler line class of parameter five in $PG(3, 3)$, then $\mathcal{L}$ consists of all of the secants to some elliptic quadric $\mathcal{O}$ plus exactly two of the tangent lines at each point of $\mathcal{O}$, or $\mathcal{L}$ is the complement of a set of this type.*

Now we construct an example. As -1 is not a square in $GF(3)$, the point set given by $F : x_0^2 + x_1^2 + x_2x_3 = 0$ is an elliptic quadric $\mathcal{O}$. Let $P = (0, 0, 0, 1) \in \mathcal{O}$. Then the tangent plane to $\mathcal{O}$ at P_1 is $\pi : x_2 = 0$. Let $\mathcal{L}_P$ be the two tangent lines $P \vee (1, 0, 0, 0)$ and $P \vee (0, 1, 0, 0)$. For each point $Q \neq P$ of $\mathcal{O}$ define $\mathcal{L}_Q$ to be the two lines tangent to $\mathcal{O}$ at Q which intersect the two lines of $\mathcal{L}_P$. This defines a set of two tangent lines to $\mathcal{O}$ at each point of $\mathcal{O}$. We claim that these sets, together with the set of secants to $\mathcal{O}$, define a Cameron–Liebler line class of parameter five in $PG(3, 3)$. The crucial point is that for each point P off $\mathcal{O}$, either all tangents to $\mathcal{O}$ on P or none of them are in $\mathcal{L}$; this can be verified by hand (there are thirty points to check) or by using a computer. Using this, one can show that the intersection property (2) is satisfied by $\mathcal{L}$ for all lines of Σ, and therefore that $\mathcal{L}$ is a Cameron–Liebler line class.

4. CONCLUDING REMARKS

In both cases $x = 4$ and $x = 5$ explored above, the existence of a Cameron–Liebler line class relied on the existence of a complete cap with certain properties. Furthermore, the construction given in [5] associates a Cameron–Liebler line class of parameter $\frac{1}{2}(q^2+1)$ to an elliptic quadric in $PG(3,q)$ for any odd prime power q. This suggests two questions: first, do there exist caps $\mathcal{K}$ other than elliptic quadrics in $PG(3,q)$ such that any point off $\mathcal{K}$ is on the same number of tangent lines to $\mathcal{K}$; and second, do such caps always give rise to Cameron–Liebler line classes? Any such cap $\mathcal{K}$ must be complete because if every point not in $\mathcal{K}$ is on the same number of tangents, then every point is on the same number of secants, and as there are more than two points on each line there will always be points which are on a secant to $\mathcal{K}$. We note that the complete caps of size $\frac{1}{2}(q^2+q+4)$ constructed from ovoids in Hirschfeld [10] do not satisfy this criterion. Furthermore, by double counting flags, one can show that if $\mathcal{K}$ is a cap of size k such that every point off $\mathcal{K}$ is on exactly a tangents to $\mathcal{K}$, then we have

$$kq(q^2+q+2-k)=(q^3+q^2+q+1-k)a.$$

Using this, one can show that the elliptic quadrics are the only caps $\mathcal{K}$ such that each point off $\mathcal{K}$ is on exactly $q+1$ tangents to $\mathcal{K}$, and that there are no caps $\mathcal{K}$ such that each point off $\mathcal{K}$ is on exactly q tangents to $\mathcal{K}$.

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